PMM U.S.S.R., Vol.44,pp.547-554 0021-8928/81/5 0547 $7.50/0

UDC 62-50

A SIMPLE QUASI-LINEAR PURSUIT PROBLEM*

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A class of differential games is delineated, in which the main pursuit operator T_t^* /1,2/ is computed analytically. The support function of set $T_t^*(M)$ is written out in explicit form. It is proved that for this class of games the optimal pursuit time coincides with the maximin pursuit time /1—5/ introduced by Kelendzheridze. A sufficient condition for the completion of pursuit in Kelendzheridze's time is obtained for a linear differential game. The closeness of this condition to the necessary condition is proved. The paper borders on the investigations in /1—10/.

1. Let the motion of a vector z in an n-dimensional Euclidean space $R^n = E$ be described by the vector differential equation

$$dz / dt = \lambda(v)z - F(u, v); \quad u \in P \subset R^p, \quad v \in Q \subset R^q \tag{1.1}$$

(u, v are control parameters, P and Q are compacta in finite-dimensional spaces, $F : P \times Q \to E$ and $\lambda : Q \to R^1$ are continuous mappings) and by the convex closed terminal set M. The statement of the pursuit problem in game (1.1), the objectives, the information available to the players have been defined in /4/. The general theory of pursuit has been constructed in /1,2/, reducing the study of pursuit problem (1.1) to the investigation of the structure of an operator $T_t^* : 2^E \to 2^E$ (in contrast to /2/ we use an asterisk instead of a tilde)

$$T_\varepsilon(X) = \bigcap_{v^* \in V} \bigcup_{\tau \in [0, \varepsilon]} \Big(f_{v^*}(\tau) X + \int_0^\tau f_{v^*}(s) P(v(s))\, ds\Big), \quad T_{\omega_t}(X) = T_{\delta_m}(T_{\delta_{m-1}}(\ldots(T_{\delta_1}(X))\ldots)); \quad T_t^*(X) = \bigcap_{\omega_t \in \Omega_t} T_{\omega_t}(X)$$

$$f_{v^*}(\tau) = \exp\Big(-\int_0^\tau \lambda(v(s))\, ds\Big); \qquad P(v) = \operatorname{conv} F(P, v)$$

Here V is the set of all measurable controls $v^* = \{v(s) \in Q,\ s \in R^1\}$, Ω_t is the set of all partitions $\omega_t = \{0 < \delta_1 < \delta_1 + \delta_2 < \ldots < \delta_1 + \ldots + \delta_m = t\}$ of interval $[0, t]$ (cf. /2,5/). The operator

$$\Theta_\varepsilon : 2^E \to 2^E; \quad \Theta_\varepsilon(X) = \bigcap_{v \in Q} \Gamma(\varepsilon, X, v), \quad \Gamma(\varepsilon, X, v) = \bigcup_{\tau \in [0, \varepsilon]} (f(\tau, v) X + \gamma(\tau, v) P(v)) \tag{1.2}$$

$$f(\tau, v) = \exp(-\tau\lambda(v)); \ \gamma(\tau, v) = \int_0^\tau f(s, v)\, ds, \quad \Theta_t^*(X) = \bigcap_{\omega_t \in \Omega_t} \Theta_{\delta_m}(\Theta_{\delta_{m-1}}(\ldots(\Theta_{\delta_1}(X))\ldots))$$

was introduced in /1,2/.

A fundamental theorem was proved in /1/: for any closed $X \subset E$ and for $t \geqslant 0$ $T_t^*(X) = \Theta_t^*(X)$.

In the present paper we present conditions sufficient for the fulfillment of the equalities

$$T_\varepsilon^*(M) = T_\varepsilon(M) = \Theta_\varepsilon(M) \tag{1.3}$$

The equalities (1.3) were first proved for the case $\lambda(v) \equiv 0$, $F(u, v) = u - v$, P and Q are convex compacta in E, in /3/.

Lemma 1. In order that (1.3) be fulfilled it is necessary and sufficient that

$$\Theta_{\varepsilon_1}(\Theta_{\varepsilon_2}(M)) = \Theta_{\varepsilon_1+\varepsilon_2}(M), \quad \forall \varepsilon_1 \geqslant 0, \quad \varepsilon_2 \geqslant 0 \tag{1.4}$$

Indeed, (1.4) follows from (1.3) and the semigroup property /2/ of operator T_t^*. Conversely, if (1.4) is fulfilled, then by induction $\Theta_{\omega_\varepsilon}(M) = \Theta_\varepsilon(M)$ for any $\omega_\varepsilon \in \Omega_\varepsilon$. It remains to make use of the fundamental theorem

$$T_\varepsilon^*(M) \subset T_\varepsilon(M) \subset \Theta_\varepsilon(M) = \bigcap_{\Omega_\varepsilon} \Theta_{\omega_\varepsilon}(M) = T_\varepsilon^*(M)$$

We observe that the inclusion of the left-hand side of (1.4) into the right always holds /3,6/.

2. Let $X \subset E$, $\psi \in E$. We set

$$W(X; \psi) = \sup_{x \in X} (x \cdot \psi); \quad K(X) = \{\psi \in E : W(X; \psi) < +\infty\}$$

*Prikl.Matem.Mekhan.,44,No.5,771-782,1980

($W(X; \psi)$ is the support function of set X). Everywhere below we assume that M and X are convex closed sets in E.

Lemma 2. For all $\varepsilon \geqslant 0$, $v \in Q$ the set $\Gamma(\varepsilon, X, v)$ is convex and closed, and

$$\Gamma(\varepsilon, X, v) = B(\varepsilon, X, v), \quad B(\varepsilon, X, v) = \operatorname{conv}(X \cup (f(\varepsilon, v)X + \gamma(\varepsilon, v)P(v)))$$

The support function $W^\varepsilon(X, v; \psi)$ of set $\Gamma(\varepsilon, X, v)$ equals

$$W^\varepsilon(X, v; \psi) = \begin{cases} +\infty, & \psi \in K^*(X) = E \setminus K(X) \\ W(X; \psi) + \gamma(\varepsilon, v)\varphi(X, v; \psi), & \psi \in K(X) \end{cases} \tag{2.1}$$

Here

$$\varphi(X, v; \psi) = \max\{0, h(X, v; \psi)\}, \quad h(X, v; \psi) = W(P(v); \psi) - \lambda(v) W(X; \psi) \tag{2.2}$$

Proof. By virtue of (1.2) it is enough to verify the inclusion $\Gamma(\varepsilon, X, v) \subset B(\varepsilon, X, v)$ and the convexity of $\Gamma(\varepsilon, X, v)$. Both these follow from the identity

$$f(\tau, v) \equiv 1 - \lambda(v)\gamma(\tau, v) \tag{2.3}$$

Indeed, for any $z = f(\tau, v)x + \gamma(\tau, v)p$ such that $\tau \in [0, \varepsilon]$, $x \in X$, $p \in P(v)$, we have from (2.3) the representation

$$z = (1-\alpha)x + \alpha(f(\varepsilon, v)x + \gamma(\varepsilon, v)p) \in B(\varepsilon, X, v); \quad 0 \leqslant \alpha = \frac{\gamma(\tau, v)}{\gamma(\varepsilon, v)} \leqslant 1$$

To verify the convexity of $\Gamma(\varepsilon, X, v)$ we take further

$$z^* = f(\tau^*, v)x^* + \gamma(\tau^*, v)p^*; \quad \tau^* \in [0, \varepsilon], \ x^* \in X, \ p^* \in P(v)$$

Since for any $\mu \in [0, 1]$ we can find (because of the continuity of $\gamma(s, v)$ with respect to $s \in [0, \varepsilon]$) $\tau_* \in [0, \varepsilon]$ such that (see (2.3))

$$\mu\gamma(\tau, v) + (1-\mu)\gamma(\tau^*, v) = \gamma(\tau_*, v), \qquad \mu f(\tau, v) + (1-\mu) f(\tau^*, v) = f(\tau_*, v)$$

we have

$$x_* = \beta x + (1-\beta)x^* \in X; \quad \beta = \mu f(\tau, v) / f(\tau_*, v) \in [0, 1], \quad p_* = \omega p + (1-\omega)p^* \in P(v); \quad \omega = \mu\gamma(\tau, v) / \gamma(\tau_*, v) \in [0, 1]$$

and, hence

$$\mu z + (1-\mu)z^* = f(\tau_*, v)x_* + \gamma(\tau_*, v)p_* \in \Gamma(\varepsilon, X, v)$$

The convexity of $\Gamma(\varepsilon, X, v)$ has been proved, and with it the first part of the lemma, from which (the closedness of $\Gamma(\varepsilon, X, v)$ follows from that of $P(v)$) follow formulas (2.1), (2.2).

Lemma 3. The set $\Theta_\varepsilon(X)$ is convex and closed. The inclusion $z \in \Theta_\varepsilon(X)$ is fulfilled if and only if

$$(z \cdot \psi) \leqslant \inf_{v \in Q} W^\varepsilon(X, v; \psi) \equiv \overline{W}^\varepsilon(X; \psi), \quad \forall \psi \in E, \quad \overline{W}^\varepsilon(X; \psi) = W(X; \psi) + \Phi(X, \varepsilon; \psi) \tag{2.4}$$

$$\Phi(X, \varepsilon; \psi) = 0, \quad \psi \in K^*(X); \quad \Phi(X, \varepsilon; \psi) = \max\{0, H(X, \varepsilon; \psi)\}, \ \psi \in K(X)$$

$$H(X, \varepsilon; \psi) = \min_{v \in Q} \gamma(\varepsilon, v) h(X, v; \psi) \tag{2.5}$$

Lemma 3 is a trivial corollary of (1.2) and Lemma 2. From (2.4) it follows that the support function $W^\varepsilon(X; \psi) = W(\Theta_\varepsilon(X); \psi)$ of set $\Theta_\varepsilon(X)$ is given /7/ by the formula

$$W^\varepsilon(X; \psi) = \inf \sum_{i=1}^{m} \overline{W}^\varepsilon(X; \psi_i) \tag{2.6}$$

where the lower bound is taken over all finite collections of vectors

$$\psi_i \in E, \quad i = 1, \ldots, m \quad (\psi_1 + \ldots + \psi_m = \psi) \tag{2.7}$$

We observe that, as follows from (2.1) and (2.6)

$$K(X) = K(\Theta_\varepsilon(X)), \quad \varepsilon \geqslant 0 \tag{2.8}$$

3. We denote

$$S^+(X) = \{\psi \in K(X) : \inf_{v \in Q} h(X, v; \psi) > 0\}; \quad S^+ = S^+(M)$$

Lemma 4. If $\psi \in K(X)$, then $\psi \notin S^+(X)$ if and only if

$$\Phi(X, \varepsilon; \psi) = 0, \quad \forall \varepsilon \geqslant 0 \tag{3.1}$$

The proof follows trivially from the inequality $\gamma(\varepsilon, v) > 0$ for $\varepsilon > 0$ and $v \in Q$.

Lemma 5. If $\psi \notin S^+(X)$, then the inequalities

$$\begin{aligned} &W^\tau(X; \psi) \equiv W(X; \psi), \quad \Phi(\Theta_\tau(X), t; \psi) \equiv 0 \\ &W(\Theta_{\omega_t}(X); \psi) \equiv W(X; \psi), \quad \Phi(\Theta_{\omega_t}(X), \tau; \psi) \equiv 0 \\ &W(\Theta_t^*(X); \psi) \equiv W(X; \psi), \quad \Phi(\Theta_t^*(X), \tau; \psi) \equiv 0 \end{aligned} \tag{3.2}$$

are fulfilled for any $t \geqslant 0$, $\tau \geqslant 0$

Proof. By virtue of (1.2)

$$X \subset \Theta_t^*(X) \subset \Theta_{\omega_t}(X) \subset \Theta_t(X) \tag{3.3}$$

Therefore, equalities (3.2) are obvious for all $\psi \in K^*(X)$. If now $\psi \in K(X)$, then from (3.3) we have $W(X;\psi) \leqslant W^\tau(X;\psi)$ which together with (2.6) yields (cf. (3.1))

$$W(X;\psi) \leqslant \overline{W^\tau}(X;\psi) = W(X;\psi) + \Phi(X,\tau;\psi) = W(X;\psi),\ \psi \notin S^+(X)$$

The first equality in (3.2) has been proved. With due regard to (2.2), (2.4), (2.5) we then obtain $\Phi(\Theta_\tau(X), t; \psi) = \Phi(X, t; \psi)$, but the latter expression equals zero by Lemma 4. The first row of equalities in (3.2) has been proved. From it, in accord with (2.8), follows the inclusion

$$S^+(\Theta_\tau(X)) \subset S^+(X) \tag{3.4}$$

for any convex closed set $X \subset E$ and for $\tau \geqslant 0$. Hence by induction

$$S^+(\Theta_{\omega_t}(X)) \equiv S^+(\Theta_{\delta_m}(\ldots(\Theta_{\delta_1}(X))\ldots)) \subset S^+(X),\ \forall \omega_t \in \Omega_t,\ t \geqslant 0 \tag{3.5}$$

in connection with which the second row of equalities in (3.2) is fulfilled. Hence, with due regard to (3.3), (2.2), (2.5) and Lemma 4 we have

$$W(X;\psi) \leqslant W(\Theta_t^*(X);\psi) \leqslant W(\Theta_{\omega_t}(X);\psi) = W(X;\psi),\quad \Phi(\Theta_t^*(X),\tau;\psi) = \Phi(X,\tau;\psi) = 0$$

4. Let

$$\lambda^* = \max_{v\in Q} \lambda(v);\quad f(\varepsilon) = \exp(-\lambda^*\varepsilon);\quad \gamma(\varepsilon) = \int_0^\varepsilon f(r)\,dr;\quad W(\psi) = \min_{v\in Q} W(P(v);\psi)$$

By $Q^+ = Q^+(M)$ we denote a subset of Q such that for each $\psi \in S^+$ we can find $\bar v \in Q^+$ satisfying the equality

$$\min_{v\in Q} h(M, v; \psi) = h(M, \bar v; \psi) \tag{4.1}$$

We assume the fulfillment of the following condition for pursuit problem (1.1).

Condition A. $\lambda(\bar v) \equiv \lambda^*$ for any $\bar v \in Q^+$.

Let $\psi \in S^+ \equiv S^+(M)$. We fix and denote by $\bar v = \bar v(\psi)$ an arbitrary vector from Q^+, given by formula (4.1). Let $X \subset E$. We set

$$W_*(\psi) = W(P(\bar v(\psi));\psi);\quad H(X;\psi) = W_*(\psi) - \lambda^* W(X;\psi)$$

$$\Phi(X;\psi) = \max\{0,\ H(X;\psi)\};\quad \Phi_\varepsilon(X;\psi) = \Phi(\Theta_\varepsilon(X);\psi)$$

Lemma 6. If condition A is fulfilled, then

$$\overline{W}^\varepsilon(\Theta_\delta(M);\psi) = W(\Theta_\delta(M);\psi) + \gamma(\varepsilon)\Phi(\Theta_\delta(M);\psi),\quad \forall\,\psi \in S^+,\quad \varepsilon \geqslant 0,\quad \delta \geqslant 0 \tag{4.2}$$

Let us first prove that if $\psi \in S^+$ and $\delta \geqslant 0$, then

$$\min_{v\in Q} h(\Theta_\delta(M), v; \psi) = h(\Theta_\delta(M), \bar v(\psi); \psi) = H(\Theta_\delta(M);\psi) \tag{4.3}$$

Indeed, by Condition A

$$W_*(\psi) - \lambda^* W^\delta(M;\psi) = h(\Theta_\delta(M), \bar v(\psi);\psi) \geqslant \min_{v\in Q}[W(P(v);\psi) - \lambda(v)W^\delta(M;\psi)] \geqslant \min_{v\in Q}[W(P(v);\psi) - \lambda(v)W(M;\psi)] +$$
$$\min_{v\in Q}\{\lambda(v)[W(M;\psi) - W(\Theta_\delta(M);\psi]\} = h(M,\bar v(\psi);\psi) + \lambda^*[W(M;\psi) - W(\Theta_\delta(M);\psi)] = H(\Theta_\delta(M);\psi)$$

Equality (4.3) has been proved. Using this equality and the property of the minimum of a product of nonnegative functions, we obtain (see (3.4))

$$\gamma(\varepsilon)H(\Theta_\delta(M);\psi) = \gamma(\varepsilon,\bar v(\psi))h(\Theta_\delta(M),\bar v(\psi);\psi) \geqslant H(\Theta_\delta(M),\varepsilon;\psi) \geqslant \min_{v\in Q}\gamma(\varepsilon,v)\cdot\min_{v\in Q}h(\Theta_\delta(M),v;\psi) =$$
$$\gamma(\varepsilon)H(\Theta_\delta(M);\psi),\ \forall\psi \in S^+(\Theta_\delta(M))$$

Consequently, $H(\Theta_\delta(M),\varepsilon;\psi) = \gamma(\varepsilon)H(\Theta_\delta(M);\psi)$ and, hence,

$$\Phi(\Theta_\delta(M),\varepsilon;\psi) = \gamma(\varepsilon)\Phi(\Theta_\delta(M);\psi),\ \forall\psi \in S^+(\Theta_\delta(M)) \tag{4.4}$$

Now if $\psi \in S^+ \setminus S^+(\Theta_\delta(M))$, then, by Lemma 4, $\Phi(\Theta_\delta(M),\varepsilon;\psi) = 0$ i.e.,

$$\min_{v\in Q} h(\Theta_\delta(M), v; \psi) \leqslant 0$$

which by virtue of (4.3) implies $H(\Theta_\delta(M);\psi) \leqslant 0$, so that $\Phi_\delta(M;\psi) = 0$. Thus we have proved that (4.4) is true for all $\psi \in S^+$. From this equality and (2.4) follows (4.2). We set $\Phi(M;\psi) = 0$, $\psi \notin S^+(M)$.

5. Lemma 7. Let $\varepsilon \geqslant 0, \psi \in K(M)$. Then for any $\Delta > 0$ we can find a collection (2.7) such that

$$0 \leqslant -W^\varepsilon(M;\psi) + \sum_{i=1}^m [W(M;\psi_i) + \gamma(\varepsilon)\Phi(M;\psi_i)] \leqslant \Delta \tag{5.1}$$

$$\Phi_\varepsilon(M;\psi) \geqslant f(\varepsilon)\sum_{i=1}^{m}\Phi(M;\psi_i) - \Delta|\lambda^*| \tag{5.2}$$

Proof. If $W^\varepsilon(M;\psi) = W(M;\psi) + \gamma(\varepsilon)\Phi(M;\psi)$, then inequality (5.1) is fulfilled for a collection (2.7) in which $m = 1, \psi_1 = \psi$. Let us verify (5.2). We estimate $H \equiv H(\Theta_\varepsilon(M);\psi)$. We have

$$H = W_*(\psi) - \lambda^* W^\varepsilon(M;\psi) = W_*(\psi) - \lambda^* W(M;\psi) - \lambda^*\gamma(\varepsilon)\Phi(M;\psi) = \tag{5.3}$$

$$H(M;\psi) - \lambda^*\gamma(\varepsilon)\Phi(M;\psi) = \begin{cases} f(\varepsilon)\Phi(M;\psi), & H(M;\psi) > 0 \\ H(M;\psi), & H(M;\psi) \leqslant 0\end{cases}$$

Hence, $\Phi_\varepsilon(M;\psi) = f(\varepsilon)\Phi(M;\psi)$ and (5.2) is proved. Consider the case

$$W^\varepsilon(M;\psi) < W(M;\psi) + \gamma(\varepsilon)\Phi(M;\psi) \tag{5.4}$$

By virtue of (3.2), (4.4) we have from this that $\Phi(M;\psi) > 0$, so that

$$\Phi(M;\psi) = H(M;\psi) \tag{5.5}$$

From (5.4), the definition of the lower bound (2.6), and formula (4.2) follows the existence of a collection (2.7) such that (5.1) and the inequality

$$\sum_{i=1}^{m}[W(M;\psi_i) + \gamma(\varepsilon)\Phi(M;\psi_i)] \leqslant W(M;\psi) + \gamma(\varepsilon)\Phi(M;\psi) \tag{5.6}$$

are fulfilled. Because of the convexity of the support function $W(M;\psi) \leqslant \sum_{i=1}^{m} W(M;\psi_i)$, , from (5.6) we have

$$\sum_{i=1}^{m}\Phi(M;\psi_i) \leqslant \Phi(M;\psi) \tag{5.7}$$

Once again we estimate H.

Case 1. $\lambda^* \geqslant 0$. From (5.3)—(5.5) follows

$$\Phi_\varepsilon(M;\psi) \geqslant H = W_*(\psi) - \lambda^* W^\varepsilon(M;\psi) \geqslant W_*(\psi) - \lambda^* W(M;\psi) - \lambda^*\gamma(\varepsilon)\Phi(M;\psi) = f(\varepsilon)\Phi(M;\psi)$$

which together with (5.7) yields (5.2).

Case 2. $\lambda^* = -|\lambda^*| < 0$. To estimate H we use (5.1), the convexity of $W(M;\psi)$ and (5.7)

$$H \geqslant W_*(\psi) + |\lambda^*|\Big\{\sum_{i=1}^{m}[W(M;\psi_i) + \gamma(\varepsilon)\Phi(M;\psi_i)] - \Delta\Big\} \geqslant W_*(\psi) +$$

$$|\lambda^*|W(M;\psi) + |\lambda^*|\gamma(\varepsilon)\sum_{i=1}^{m}\Phi(M;\psi_i) - \Delta|\lambda^*| = \Phi(M;\psi) - \lambda^*\gamma(\varepsilon)\sum_{i=1}^{m}\Phi(M;\psi_i) - \Delta|\lambda^*| \geqslant f(\varepsilon)\sum_{i=1}^{m}\Phi(M;\psi_i) - \Delta|\lambda^*|$$

Hence follows (5.2).

6. Theorem 1. If Condition A is fulfilled for problem (1.1), then (1.3) is fulfilled for any $\varepsilon \geqslant 0$

Proof. By virtue of Lemmas 1 and 3 it suffices (see Section 1) to verify the inequality

$$\overline{W}^{\varepsilon_1}(\Theta_{\varepsilon_2}(M);\psi) \geqslant W^{\varepsilon_1+\varepsilon_2}(M;\psi);\quad \psi \in E,\quad \varepsilon_1 \geqslant 0,\quad \varepsilon_2 \geqslant 0 \tag{6.1}$$

If $\psi \in K^*(M)$ or $\psi \in K(M)\setminus S^+$, then (6.1) follows from (2.8) or (3.2), respectively. Now let $\psi \in S^+, \Delta > 0$. In accord with Lemma 7 a collection (2.7) exists such that inequalities (5.1) and (5.2) are fulfilled for $\varepsilon = \varepsilon_2$, combining which we obtain

$$\overline{W}^{\varepsilon_1}(\Theta_{\varepsilon_2}(M);\psi) = W^{\varepsilon_2}(M;\psi) + \gamma(\varepsilon_1)\Phi_{\varepsilon_2}(M;\psi) \geqslant \sum_{i=1}^{m}[W(M;\psi_i) + \tag{6.2}$$

$$\gamma(\varepsilon_2)\Phi(M;\psi_i)] - \Delta + \gamma(\varepsilon_1)f(\varepsilon_2)\sum_{i=1}^{m}\Phi(M;\psi_i) - \gamma(\varepsilon_1)\Delta|\lambda^*| =$$

$$\sum_{i=1}^{m}\{W(M;\psi_i) + \gamma(\varepsilon_1+\varepsilon_2)\Phi(M;\psi_i)\} - \Delta(1 + \gamma(\varepsilon_1)|\lambda^*|) \geqslant W^{\varepsilon_1+\varepsilon_2}(M;\psi) - \Delta(2 + f(\varepsilon_1))$$

Here we have used relations (4.2), (2.6) and the identity $f(\tau, v)\gamma(s, v) + \gamma(\tau, v) \equiv \gamma(\tau + s, v)$. Since the quantity $\Delta > 0$ in (6.2) is arbitrary, inequality (6.1) has been proved and with it the theorem.

7. By $Q^{++} = Q^{++}(M)$ we denote a subset of Q such that for each $\psi \in S^+$ we can find $\bar{v} \in Q^{++}$ for which $W(P(\bar{v});\psi) = W(\psi)$.

Condition B. $0 \in M$ (0 is the null vector in E); $\lambda(\bar{v}) \equiv \lambda^*$ for any $\bar{v} \in Q^{++}$.

Lemma 8. If Condition B is fulfilled for game (1.1), then Condition A is fulfilled. In this case $W_*(\psi) = W(\psi)$.

8. Let a pursuit problem be described by the equation /4/

$$dz/dt = Cz - F(u, v) \tag{8.1}$$

where C is a constant nth-order square matrix, $F(u, v)$, P, Q and M satisfy the requirements in Sect.1 for problem (1.1).

Theorem 2. Equalities (1.3) are fulfilled for problem (8.1) if matrix $C = \lambda^* I$, λ^* is a real constant, I is the nth-order unit matrix, function $F(u, v)$ is continuous on $P \times Q$, P and Q are compacts in finite-dimensional spaces, M is a convex closed set.

Proof. If the hypotheses of Theorem 2 are satisfied, then problem (8.1) turns into problem (1.1) in which $\lambda(v) \equiv \lambda^*$, $v \in Q$, in connection with which Condition A is fulfilled. It remains to apply Theorem 1.

9. Theorem 3. If in problem (8.1) each nonzero vector $\psi \in K(M)$ is an eigenvector of matrix C^* (the operator adjoint to C), the equalities (1.3) are fulfilled.

Proof. Since $K(M)$ is a convex cone /7/, a single real λ^* exists such that $C^*\psi = \lambda^*\psi$ for all $\psi \in K_0$, where K_0 is a subspace, being the linear hull of $K(M)$. If the dimension $n_* = \dim K_0 = n$, then matrix C^*, and with it also C, has the form $\lambda^* I$, where I is the nth-order unit matrix. It remains to make use of Theorem 2. Now let $n_* < n$. By N_0 we denote the orthogonal complement to K_0 in E and by π we denote the operator of orthogonal projection onto K_0. Then the following lemma is valid.

Lemma 9. Set M can be represented as

$$M = N_0 + M_*; \quad M_* = \pi M \tag{9.1}$$

Proof. At first we verify the set $m_0 + N_0$ is contained in M for any $m_0 \in M$. To the contrary suppose that we can find $n_0 \in N_0$ such that $m_0 + n_0 \notin M$; then by the separability theorem /7/ there exists $\psi \in K(M)$ such that $(\psi \cdot (m_0 + n_0)) > (\psi \cdot m)$ for every $m \in M$. Taking $m = m_0$ and recalling that $(\psi \cdot n_0) = 0$, we arrive at a contradiction. To prove (9.1) it remains to make use of the chain of inclusions

$$N_0 + M_* = N_0 + M \subset M \subset N_0 + M_*$$

We complete the theorem's proof. We set $z_* = \pi z$; $z^* = z - \pi z$; $F_*(u, v) = \pi F(u, v)$; $C_* = \pi C$. By virtue of (9.1), $z \in M$ if and only if $z_* \in M_*$. Further, since subspace K_0 is invariant relative to operator C^*, we have that N_0 is invariant relative to C, so that $C_* z^* \equiv 0$. In addition, for any $\psi \in K_0$ we have

$$(\psi \cdot [C_* z_* - \lambda^* z_*]) = (\psi \cdot C z_*) - \lambda^* (\psi \cdot z_*) = (C^*\psi \cdot z_*) - \lambda^* (\psi \cdot z_*) = 0$$

Consequently, $C_* z_* = \lambda^* z_*$. Therefore, applying operator π to (8.1), we obtain

$$dz_* / dt = \pi (dz / dt) = C_* (z_* + z^*) - F_* (u, v) = \lambda^* z_* - F_* (u, v) \tag{9.2}$$

Thus, under the hypotheses of Theorem 3 game (8.1) is equivalent to game (9.2) with terminal set M_*. Since game (9.2) already satisfies the hypotheses of Theorem 2, Theorem 3 is proved.

Let us now prove a theorem that in some sense is the converse to Theorem 3.

Theorem 4. Let a matrix C and a convex closed terminal body M /8/ be such that we can find a vector $\varphi_0 \in K(M)$, $|\varphi_0| = 1$, that is not an eigenvector of operator C^*. Then for any sufficiently small $\theta_0 > 0$ there exist the spheres $P = a + (\rho + \sigma)S$, $Q = \rho S$ ($\rho > 0$, $\sigma > 0$; S is the closed unit sphere in E with center at the origin; a is a constant vector) and the function $F(u, v) = u - v$ such that in problem (8.1) the sets $T_\varepsilon(M)$ and $T_\varepsilon^*(M)$ do not coincide for some $\varepsilon \in (0, \theta_0)$ (here, as in Theorem 3, we use the general definition /2/ of operators T_ε and T_ε^*).

The theorem is proved in several stages. By $v(\psi)$ and $\omega(\psi)$ we denote (and use subsequently) the vectors occurring in the equalities.

$$v(\psi) \in Q, \quad (\psi \cdot v(\psi)) = W(Q; \psi); \quad \omega(\psi) \in \Omega, \quad (\psi \cdot \omega(\psi)) = W(\Omega; \psi)$$

Lemma 10. Let Q and Ω be convex compact bodies not containing segments on the boundary, where only one support hyperplane passes through each point of the boundary of Ω. Then for any $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that the inclusion $v + \omega(\psi) + \delta_0\varphi \in P = Q + \Omega$ is fulfilled for all $\varphi, \psi \in S$, $|\psi| = 1$, and for all $v \in Q$ satisfying the inequality $|v(\psi) - v| \geqslant \varepsilon_0$.

Proof. By the lemma's hypotheses the vectors $v(\psi)$ and $\omega(\psi)$ are unique. The subsequent argument is by contradiction. Let $\varepsilon_0 > 0$ exists such that for any positive integer n we can find ψ_n, $|\psi_n| = 1$, $\varphi_n \in S$; $v_n \in Q$, $|v(\psi_n) - v_n| \geqslant \varepsilon_0$, such that $v_n + \omega(\psi_n) + \frac{1}{n}\varphi_n \notin P$, i.e., the inequality

$$\left(\psi^n \cdot \left[v(\psi^n) + \omega(\psi^n) - v_n - \omega(\psi_n) - \frac{1}{n}\varphi_n\right]\right) < 0 \tag{9.3}$$

is fulfilled for some $\psi^n \in E$, $|\psi^n| = 1$. Passing, if necessary, to a subsequence, we can take it that

$$\psi_n \to \psi_0, \quad \psi^n \to \psi^0, \quad \varphi_n \to \varphi_0, \quad |\psi_0| = |\psi^0| = 1 \geqslant |\varphi_0|; \quad v_n \to v_0 \in Q,) \quad n \to \infty$$

Since (because of the absence of segments on the boundary) the functions $v(\psi)$ and $\omega(\psi)$ are continuous /7/.

$$v(\psi_n) \to v(\psi_0), \quad v(\psi^n) \to v(\psi^0), \quad \omega(\psi_n) \to \omega(\psi_0), \quad \omega(\psi^n) \to \omega(\psi^0), \quad n \to \infty$$

Passing to the limit, from (9.3) we obtain

$$(\psi^0 \cdot [v(\psi^0) - v_0]) + (\psi^0 \cdot [\omega(\psi^0) - \omega(\psi_0)]) \leqslant 0 \tag{9.4}$$

$$|v(\psi_0) - v_0| \geqslant \varepsilon_0 \tag{9.5}$$

Noting that each of the two terms in the left-hand side of (9.4) are nonnegative, we conclude that $v_0 = v(\psi^0)$; $\omega(\psi^0) = \omega(\psi_0)$. Since only one support hyperplane to Ω passes through the point $\omega(\psi_0)$, we have that $\psi_0 = \psi^0$, so that $v_0 = v(\psi_0)$, but this contradicts (9.5).

10. Everywhere in this section we assume that in game (8.1) M is a convex closed body, $F(u, v) = u - v$, $P = Q + \Omega$, where Q and Ω satisfy the hypotheses of Lemma 10. For any $z_0 \in E$ we denote by $t(z_0)$ the earliest instant $t \geqslant 0$ for which the inclusion $z_0 \in T_t^*(M)$ is fulfilled. It is well known /9/ that in this case the instant $t(z_0)$ can be defined also as the earliest instant $t \geqslant 0$ for which the inclusion

$$\Phi(t) z_0 \in M + \int_0^t \Phi(r) \Omega dr; \quad \Phi(t) \equiv \exp(tC) \tag{10.1}$$

is fulfilled. Using the notation of Lemma 10, we assume

$$v(r, \psi) = v(w(r, \psi)), \quad \omega(r, \psi) = \omega(w(r, \psi)), \quad w(r, \psi) = \Phi^*(r)\psi / |\Phi^*(r)\psi|, \quad \Phi^*(r) \equiv \exp(rC^*)$$

Then vectors $m_0 \in M$ and $\psi_0 \in E$, $|\psi_0| = 1$ exist such that

$$\Phi(t(z_0)) z_0 = m_0 + \int_0^{t(z_0)} \Phi(r) \omega(r, \psi_0) dr \tag{10.2}$$

$$\Phi(t) z_0 \notin M + \int_0^t \Phi(r) \Omega dr, \quad t \in [0, t(z_0)) \tag{10.3}$$

Lemma 11. For the fulfillment of the equality $T_\varepsilon(M) = T_\varepsilon^*(M)$, $\varepsilon \in [0, t(z_0)]$ it is necessary that the condition (int is the symbol for the interior of a set)

$$\Phi(t) z_0 + \int_0^t \Phi(t - r) v(t(z_0) - r, \psi_0) dr \notin \operatorname{int}\left[M + \int_0^t \Phi(r) P dr\right] \tag{10.4}$$

be fulfilled for any $t \in [0, t(z_0))$.

Proof (by contradiction). Let $\tau \in [0, t(z_0))$ exist such that

$$\Phi(\tau) z_0 + \int_0^\tau \Phi(\tau - r) v(t(z_0) - r, \psi_0) dr \in \operatorname{int}\left[M + \int_0^\tau \Phi(r) P dr\right] \tag{10.5}$$

This signifies that we can find $q_0 > 0$ such that for each measurable control $v(t) \in Q$, $t \in [0, \tau]$, satisfying the inequality

$$\int_0^\tau |v(t(z_0) - r, \psi_0) - v(r)| dr \leqslant q_0 \tag{10.6}$$

we can find a measurable control $u(t) \in P$, $t \in [0, \tau]$, such that

$$z(\tau) = \Phi(\tau) z_0 - \int_0^\tau \Phi(\tau - r)[u(r) - v(r)] dr \in M \tag{10.7}$$

We assume

$$\varepsilon_0 = \frac{q_0}{2\tau}, \quad k_0 = \frac{q_0}{4Q^*}, \quad Q^* = \max_{v \in Q, \omega \in \Omega} \{|v| + |\omega|\}$$

Let a number $\delta_0 > 0$ correspond to ε_0 by virtue of Lemma 10. Since M is a body, there exist a vector φ_* and a number $\mu_0 > 0$ such that /8/

$$|\varphi_*| \neq 0; \quad |\Phi^{-1}(r) \varphi_*| \leqslant 1, \; 0 \leqslant r \leqslant t(z_0), \quad m_* + \mu_0 S \subset M, \; m_* = m_0 - k_0 \delta_0 \varphi_* \tag{10.8}$$

We set $B_1 = |m_*| + 1 + Q^* / B_2$; $B_2 = \|C\| + 1$; $\|C\|$ is the norm of matrix C. We select a number $s_0 \in (\tau, t(z_0))$ such that

$$B_1 [\exp(B_2 |t(z_0) - s_0|) - 1] < \mu_0 \tag{10.9}$$

We now consider an arbitrary measurable control $v_* = \{v(t) \in Q, \; t \in [0, s_0]\}$ not satisfying (10.6). This signifies that a measurable set $V(v_*) \subset [0, \tau]$ exists such that $\operatorname{mes} V(v_*) = k_0$, $|v(t(z_0) - r, \psi_0) - v(r)| \geqslant \varepsilon_0$, $r \in V(v_*)$. Having defined $v(r) \equiv v(t(z_0) - r, \psi_0)$, $r \in (s_0, t(z_0)]$, we assume

$$u(r) = \begin{cases} v(r) + \omega(t(z_0) - r, \psi_0) + \delta_0 \Phi^{-1}(t(z_0) - r) \varphi_*, & r \in V(v_*) \\ v(r) + \omega(t(z_0) - r, \psi_0), & r \in [0, t(z_0)] \setminus V(v_*) \end{cases}$$

(the possibility for such a choice of $u(r) \in P$ follows from Lemma 10 and (10.8)). Then for such a pair of controls $u(r)$ and $v(r)$ (cf. (10.2))

$$z(t(z_0)) = \Phi(t(z_0))z_0 - \int_0^{t(z_0)} \Phi(t(z_0) - r)[u(r) - v(r)]\,dr = m_* \tag{10.10}$$

Having denoted $l(r) = z(r) - m_*$, we have

$$|l(r)| = |l(t(z_0)) - \int_r^{t(z_0)} [Cz(\theta) - \omega(t(z_0) - \theta, \psi_0)]\,d\theta| \leqslant B_2 \int_r^{t(z_0)} (|l(\theta)| + B_1)\,d\theta, \quad r \in [s_0, t(z_0)]$$

So that by Gronwall's lemma and formula (10.9)

$$|l(s_0)| \leqslant B_1 [\exp(B_2 [t(z_0) - s_0]) - 1] < \mu_0$$

By virtue of (10.8) this signifies that $z(s_0) \in M$. Hence from (10.7) it follows that $z_0 \in T_{s_0}(M)$ (see /2/ for the definition of T_ε); however, by virtue of (10.3), $z_0 \notin T_{s_0}{}^*(M)$. A contradiction. The lemma has been proved.

11. We complete the proof of Theorem 4. We consider the analytic functions

$$\Lambda(r) = \frac{1 - e^{-r}}{r} = 1 - \frac{r}{2!} + \frac{r^2}{3!} - \ldots; \quad Y(r) = \frac{r}{1 - e^{-r}} = 1 + \frac{r}{2} + \frac{r^2}{12} + \ldots$$

whose radii of convergence are $+\infty$ and 2π, respectively. If A is an arbitrary nth-order matrix, $\|A\| < 2\pi$, , then the matrix-valued functions $\Lambda(A)$ and $Y(A)$ exist and satisfy the relations

$$\Lambda(A) \cdot Y(A) = I; \quad t\Lambda(tA) = \int_0^t \exp(-rA)\,dr, \quad t \geqslant 0$$

in connection with which the inverse operator

$$\left[\int_0^t \exp(-rA)\,dr\right]^{-1} = \frac{1}{t} Y(tA) \equiv R(t, A) \tag{11.1}$$

exists for $0 < t\|A\| < 2\pi$ Let m_0 be a fixed point of set M such that $(m_0 \cdot \varphi_0) = W(M; \varphi_0)$. Since M is a body, a vector φ_*, $|\varphi_*| = 1$, and a number $\mu_0 > 0$ exist such that

$$m_0 - \lambda\varphi_* \in \operatorname{int} M, \quad \lambda \in (0, \mu_0) \tag{11.2}$$

The vector φ_0 is not an eigenvector of operator $D = C^*$. Therefore, the number $\alpha = |D\varphi_0|^2 - (\varphi_0 \cdot D\varphi_0)^2 > 0$.

Now let $\theta_i \geqslant 0$ $(i = 1, 2, 3, 4)$ be arbitrary numbers satisfying the relations

$$\theta_4 \in \left(0, \min\left\{1, \frac{\pi}{\|C\|}\right\}\right), \quad \theta_2 > 0, \quad \theta_3 > 0, \quad \theta_2 + \theta_3 = \theta_4, \quad \theta_1 \in [0, \theta_4] \tag{11.3}$$

We set $G(x, y) = x\Lambda(xC) \cdot R(y, C)$ and consider the expressions

$$g(\theta_1, \theta_2, \theta_3) = \left(\varphi_0 \cdot G(\theta_1, \theta_2)\left[\chi(0, \theta_3) - \int_0^{\theta_3} \Phi(r) w(\theta_2 + r, \varphi_0)\,dr\right]\right)$$

$$\chi(x, y) = \int_x^y \Phi(r) w(r, \varphi_0)\,dr, \quad \xi(\theta_1, \theta_2) = (\varphi_0 \cdot G(\theta_1, \theta_2)\varphi_*)$$

$$\varkappa(\theta_1, \theta_2, \theta_3) = (\varphi_0 \cdot [\chi(\theta_4 - \theta_1, \theta_4) - G(\theta_1, \theta_2)\chi(\theta_3, \theta_4)])$$

The Taylor expansions of these expressions in powers of $\theta_1, \theta_2, \theta_3$ lead to the following estimates (N denotes a constant depending only on matrix C and not depending on θ_i $(i = 1, \ldots, 4)$):

$$g(\theta_1, \theta_2, \theta_3) \geqslant \theta_1\theta_3\left(\frac{\alpha}{2}\theta_1 - N\theta_4^2\right); \quad \xi(\theta_1, \theta_2) \leqslant N\theta_1\theta_2^{-1}, \quad \varkappa(\theta_1, \theta_2, \theta_3) \geqslant \theta_1\left(\frac{\alpha}{12}(\theta_2 - \theta_1)(\theta_2 - 2\theta_1) - N\theta_4^2|\theta_2 - \theta_1|\right) \tag{11.4}$$

Now let $p > 0$, $q > 0$, $\nu > 0$ be arbitrary real numbers satisfying together with θ_i $(i = 1, \ldots, 4)$ the inequalities

$$\nu \leqslant p\theta_2\theta_4^2, \quad \theta_4 < \frac{\alpha}{200N}, \quad \frac{1}{2}\theta_4 < \theta_2 < \frac{2p}{q} < \frac{\alpha}{100N} \tag{11.5}$$

From (11.4), (11.5) we have the estimate

$$\eta(p, q, \nu, \theta_2, \theta_3; r) > 0, \quad r \in (0, \theta_4] \tag{11.6}$$

for the expression $\eta(p, q, \nu, \theta_2, \theta_3; r) = \frac{1}{r}\{p\theta_3^{-1}g(r, \theta_2, \theta_3) + q\varkappa(r, \theta_2, \theta_3) - \nu\xi(r, \theta_2)\}$.

Let us now consider a game (8.1) in which

$$F(u, v) = u - v, \quad P = Q + \Omega, \quad Q = \rho S, \quad \Omega = a + \sigma S \tag{11.7}$$

$$a = Cm^* + a_*, \quad m^* = m_0 + \sigma\chi(0, \theta_0), \quad a_* = R(\theta_0 - \tau, C) \times \left\{-\nu\varphi_* + \rho\left[\chi(0, \tau) - \int_0^\tau \Phi(r) w(\theta_0 - \tau + r, \varphi_0)\,dr\right] - \sigma\chi(\tau, \theta_0)\right\}$$

and the constants $\rho > 0$, $\sigma > 0$, $\theta_0 > 2\tau > 0$, $\nu > 0$ satisfy the inequalities

$$\theta_0 < \min\{1, \pi \| C \|^{-1}, \alpha (200N)^{-1}\}, \quad \frac{100N}{\alpha} < \frac{\sigma}{2\rho\tau} < \theta_0^{-1}, \quad \nu < \frac{\sigma\theta_0^4}{2} \tag{11.8}$$

We take the point $z_0 = m^* + \theta_0 \Lambda(\theta_0 C) a_*$ and we verify that the equality $t(z_0) = \theta_0$ is fulfilled for this point and that at instant τ the inclusion (10.5) is fulfilled with $\psi_0 = \varphi_0$, which by Lemma 11 completes the proof of Theorem 4. It is easily verified that (cf. (10.2))

$$\Phi(\theta_0) z_0 = m_0 + \int_0^{\theta_0} \Phi(r) \omega(r, \varphi_0) dr; \quad \omega(r, \varphi_0) \equiv a + \sigma w(r, \varphi_0)$$

Let us prove (10.3) for all $t \in [0, \theta_0)$. For this it is enough to verify the inequality

$$\Delta(t) \equiv \left(\varphi_0 \cdot \left\{\Phi(t) z_0 - m_0 - \int_0^t \Phi(r) \omega(r, \varphi_0) dr\right\}\right) > 0, \quad \forall t \in [0, \theta_0)$$

Simplifying the expression for $\Delta(t)$, we obtain, using (11.7), (11.8), (11.5), (11.3), (11.6),

$$\Delta(t) = \left(\varphi_0 \cdot \int_0^{\theta_0 - t} \Phi(-r) a_* dr + \sigma \chi(t, \theta_0)\right) = (\theta_0 - t) \times \eta(\rho\tau, \sigma, \nu, \theta_0 - \tau, \tau; \theta_0 - t) > 0$$

The equality $t(z_0) = \theta_0$ has been proved.

To verify (10.5) with $\psi_0 \equiv \varphi_0$ we make use of (11.7), (11.1), (11.2)

$$\Phi(\tau) z_0 + \int_0^\tau \Phi(\tau - r) v(t(z_0) - r, \varphi_0) dr = \Phi(\tau) z_0 + \rho \int_0^\tau \Phi(\tau - r) w(\theta_0 - r, \varphi_0) dr = m_0 - \nu\varphi_* +$$
$$\int_0^\tau \Phi(r)[a + (\rho + \sigma) w(r, \varphi_0)] dr \in \operatorname{int}\left[M + \int_0^\tau \Phi(r) P dr\right]$$

The theorem has been proved.

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Translated by N.H.C.